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Distribution of transverse distances in directed animals

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Abstract

We relate $\phi(\mathbf{x}, s)$, the average number of sites at a transverse distance $\mathbf{x}$ in the directed animals with s sites in d transverse dimensions, to the two-point correlation function of a lattice gas with nearest neighbour exclusion in d dimensions. For large s, $\phi(\mathbf{x}, s)$ has the scaling form $\frac{s}{R_s^d} f(|\mathbf{x}|/R_s)$, where R_s is the root-mean square radius of gyration of animals of s sites. We determine the exact scaling function for $d = 1$ to be $f(r) = \frac{\sqrt{\pi}}{2\sqrt{3}}\mathrm{erfc}(r/\sqrt{3})$. We also show that $\phi(\mathbf{x} = 0, s)$ can be determined in terms of the animal number generating function of the directed animals.

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The directed animals (DA) problem describes the large-scale geometrical structure of subcritical directed percolation clusters, and qualitatively models diverse situations such as trees, river networks and dilute polymers in a flowing solvent [1, 2]. Formally, it corresponds to the $p \to 0$ limit of the directed percolation problem [3]. The imposed directionality changes the universality class, and the large-scale structure of directed animals is different from isotropic animals. The enumeration of directed site animals in $d + 1$ dimensions is related to hard-core lattice gas (HCLG) at negative activity with repulsive interactions in d dimensions and the Yang–Lee edge problem in d dimensions [4–7]. This equivalence helps to obtain several exact results for the directed animals problem [8–11]. In particular, the critical exponents θ and the transverse size exponent $\nu_\perp$ are known exactly in $d = 0$, 1 and 2 [8]. For reviews of the available exact results, see [12]. The problem is also related to a recently studied model of quantum gravity [13], and there is an unexpected relation between the number of distinct eigenvalues for the Potts model partition function on strips of width w and the number of directed animals with w sites in two dimensions [14].

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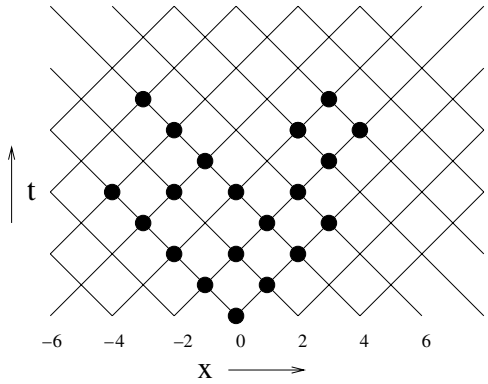

Figure 1. Schematic figure of a directed animal of size 20 on a square lattice.

In this paper, we extend the known relation between the directed site animals enumeration (DSAE) problem to hardcore lattice gases at negative activity to obtain the average number of sites at a given transverse distance **x** from the origin for $(d+1)$-dimensional directed animals from the density–density correlation function of the lattice gas in d dimensions. For $d = 1$, using the exact generating function for this correlation function, we determine the scaling form for the average number of sites at a given transverse distance in a two-dimensional directed animal having s sites, for large s. In this case, the average transverse size of the animal scales as $s^{\nu_\perp}$, where $\nu_\perp$ is known to be $\frac{1}{2}$ for $d = 1$. For large s, the average value of the qth transverse moment $\langle |x|^q \rangle$ varies as $R_s^q C_q$, where R_s is the root mean square radius of gyration of animals of s sites. Using the exact scaling function we are able to determine the universal constants C_q for all q.

We consider directed animals on a square lattice. Each site (x, t) has two bonds directed outwards towards the sites $(x+1, t+1)$ and $(x-1, t+1)$ (figure 1). A directed site animal $\mathcal{A}$ rooted at the origin is a set of occupied sites including the origin, such that for each occupied site (x, t) other than the origin, at least one of the two sites $(x-1, t-1)$ and $(x+1, t-1)$ is also occupied. Figure 1 shows a directed animal on a square lattice in two dimensions. The number of sites in $\mathcal{A}$ will be denoted by $|\mathcal{A}|$ or s. We define $n(x|\mathcal{A})$ as the number of sites of $\mathcal{A}$ having the transverse coordinate x. In figure 1, $n(x|\mathcal{A})$ takes the values 1, 3 and 0 for x equal to -4, 2, 5, respectively, with $|\mathcal{A}| = 20$. We shall define the radius of gyration, of $\mathcal{A}$ as $\sqrt{\frac{1}{s}\sum x^2}$, where the sum is over all sites of the animal. The animal shown in figure 1 has a squared radius of gyration $21/4$.

We define $A(y)$ as $y^{|\mathcal{A}|}$ summed over all animals $\mathcal{A}$ rooted at the origin. If A_s is the number of distinct animals having s sites, it is easy to see that

$$A(y) = \sum_{\mathcal{A}} y^{|\mathcal{A}|} = \sum_{s=1}^{\infty} A_s y^s. \tag{1}$$

For large s, A_s varies as $\lambda^s s^{-\theta}$, where θ is a critical exponent. The radius of gyration R_s is expected to vary as $s^{\nu_\perp}$, where the exponent $\nu_\perp$ is related to the animals number exponent θ by the hyperscaling relation $\theta = d\nu_\perp$ [15].

For a directed animal of size s we define the generating function $\Psi(\mathbf{x}, y)$ as

$$\Psi(\mathbf{x}; y) = \sum_{\mathcal{A}} n(\mathbf{x}|\mathcal{A}) y^{|\mathcal{A}|} \tag{2}$$

where the summation over $\mathcal{A}$ is the summation over all animal configurations. This can be written as

$$\Psi(\mathbf{x}; y) = \sum_s \phi(\mathbf{x}, s) A_s y^s \tag{3}$$

where $\phi(\mathbf{x}, s)$ is the value of $n(\mathbf{x}, \mathcal{A})$ averaged over all animals $\mathcal{A}$ of size s.

The DSAE problem in $d + 1$ dimensions is related to the time development of thermal relaxation of a hard core lattice gas (HCLG) with nearest neighbour exclusion on the d-dimensional lattice [5]. On a d-dimensional body-centred hypercubical lattice (a d-dimensional body-centred hypercubical lattice is a hypercubical lattice having 2^{d-1} bonds going outwards from each site), the dynamics of the lattice gas is defined as follows: the evolution is stochastic discrete-time Markovian. At odd (even) times, all the odd (even) sites are examined in parallel, and if a site has all neighbours empty, its occupation number is set to 1 with probability p, and to 0 with probability $(1 - p)$. The rates of transitions satisfy a detailed balance condition corresponding to the Hamiltonian

$$H = +\infty \sum_{\langle ij \rangle} n_i n_j - (\ln z) \sum_i n_i \tag{4}$$

with $z = p/(1 - p)$.

Let $\rho(p)$ be the average density of particles in the steady state of this system. In [5], it was shown that we have

$$A(y) = -\rho(p = -y). \tag{5}$$

For the $d = 1$ nearest-neighbour-exclusion lattice gas, it is straightforward to determine the average density corresponding to chemical activity $p/(1 - p)$. This for the square lattice gives

$$A(y) = -\frac{1}{2}\left[1 - \sqrt{\left(\frac{1+y}{1-3y}\right)}\right]. \tag{6}$$

The animal numbers A_s, which are the coefficients in the Taylor expansion of $A(y)$ for the square lattice, can be written as [8]

$$A_s = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}(1 + \cos\theta)(1 + 2\cos\theta)^{s-1}. \tag{7}$$

For large s, $A_s \sim \frac{1}{\sqrt{3\pi}} 3^s s^{-\frac{1}{2}}$, which shows that in this case $\lambda = 3$, and $\theta = 1/2$.

The derivation of equation (5) is easily generalized to the case where the values of p at different sites are different. Let the probability that site i is occupied at time $t + 1$ given that all its neighbours are unoccupied at time t be p_i. Then the rates of this process still satisfy the detailed balance condition corresponding to the Hamiltonian

$$H = +\infty \sum_{\langle ij \rangle} n_i n_j - \sum_i (\ln z_i) n_i \tag{8}$$

where $z_i = p_i/(1 - p_i)$. The probability that site i is occupied in the steady state depends on the p_j for all sites j, and will be denoted by $\rho_i(\{p_j\})$. In the corresponding DA problem, we have to define the weight of an animal $\mathcal{A}$ as the product of weights of all occupied sites, the weight corresponding to a site with $\mathbf{x}$-coordinate j being y_j. Then define $A_i(\{y_j\})$ as the sum of weights of all animals rooted at i. Then, clearly $A_i(\{y_j\})$ is a formal power series in the variables $\{y_j\}$. If all $y_j = y$, this becomes independent of i, and reduces to the function $A(y)$. For unequal y_j's, equation (5) becomes

$$A_i(\{y_j\}) = -\rho_i(\{p_j = -y_j\}). \tag{9}$$

Applying the operator $y_{\mathbf{x}}\frac{\partial}{\partial y_{\mathbf{x}}}$ on the weight of any particular animal gives us the weight multiplied by the number of occupied sites with transverse coordinate $\mathbf{x}$ in the animal. Thus, clearly, we get

$$y_{\mathbf{x}}\frac{\partial}{\partial y_{\mathbf{x}}}A_0(\{y_j\})\bigg|_{\{y_j=y\}} = \Psi(\mathbf{x};\, y). \tag{10}$$

Let $\Omega(\{z_j\})$ be the grand partition function for the HCLG given by the Hamiltonian in equation (8) with $z_j = p_j/(1-p_j)$. Then, $\Omega(\{z_j\})$ is a linear function of each of the variables z_j. Let η_j be the indicator variable taking value 1 if the site j is occupied, and 0 if not. The density–density correlation function of the gas $G(\mathbf{i}, \mathbf{k})$ is defined as

$$G(\ell, \mathbf{k}; \{z_j\}) = \langle \eta_{\mathbf{k}}\eta_{\mathbf{i}}\rangle - \langle \eta_{\mathbf{k}}\rangle\langle \eta_{\mathbf{i}}\rangle = z_{\mathbf{k}}\frac{\partial}{\partial z_{\mathbf{k}}}\rho_{\mathbf{i}}(\{z_j\}). \tag{11}$$

When $z_j = z$ for all j, the correlation function depends only on $(\mathbf{i}-\mathbf{k})$ and hence can be written as $G(\mathbf{i}-\mathbf{k};\, z)$.

Now, using equation (9), we get

$$\Psi(\mathbf{x};\, y) = -\frac{1}{1+y}G\left(\mathbf{x};\, z = -\frac{y}{1+y}\right). \tag{12}$$

This equation is the main result of this paper, and equates the pair correlation function $G(\mathbf{x};\, z)$ of the HCLG with the function $\Psi(\mathbf{x};\, y)$ which gives the density profile of the DA problem.

For the special case $d = 1$, it is a simple exercise to calculate $G(x;\, z)$ explicitly, using transfer matrix methods [16]. This gives

$$G(x;\, z) = \frac{z}{1+4z}\left[\frac{1-\sqrt{1-4z}}{1+\sqrt{1+4z}}\right]^{|x|} \tag{13}$$

and we get the explicit expression for $\Psi(x;\, y)$ on the square lattice to be

$$\Psi(x;\, y) = \frac{y}{(1+y)(1-3y)}\left[1-\sqrt{\frac{1-3y}{1+y}}\right]^{|x|}\left[1+\sqrt{\frac{1-3y}{1+y}}\right]^{-|x|}. \tag{14}$$

This determines the density profile in the constant fugacity ensemble, where an animal having s sites has weight y^s. However, it is more instructive to look at the profile in the constant-s ensemble. This is obtained by looking at the Taylor coefficient of y^s in the above equation. This can become rather messy. However, the behaviour for large s can be determined easily.

In general, for any dimension d, for large s, $\phi(\mathbf{x}, s)$ has the scaling form

$$\phi(\mathbf{x}, s) \sim \frac{s}{R_s^d} f\left(\frac{|\mathbf{x}|}{R_s}\right). \tag{15}$$

Usually the argument of the scaling function is defined only up to a multiplicative constant. We have made a specific choice for this by using the variable as $|\mathbf{x}|/R_s$. The normalization of scaling function $f(r)$ is chosen such that it satisfies

$$\int_{-\infty}^{\infty} \mathrm{d}^d\mathbf{x}\, f(|\mathbf{x}|) = 1 \tag{16}$$

$$\int_{-\infty}^{\infty} \mathrm{d}^d\mathbf{x}\, |\mathbf{x}|^2 f(|\mathbf{x}|) = 1. \tag{17}$$

In the DA problem, as the number of animals increases as λ^s, the series expansions for $\Psi(\mathbf{x}; y)$ or $A(y)$ in powers of y converge for $y < y_c = 1/\lambda$. For y near y_c, the singular part of the function $A(y)$ varies as $(1 - y\lambda)^{\theta-1}$. For the HCLG problem, this corresponds to a singularity in the series for the density ρ in powers of activity for the activity $z_{\mathrm{LY}} = -(1+\lambda)^{-1}$ (equation (12)). This singularity on the negative real line is the Lee–Yang edge singularity for this problem.

For z near the critical value $z = z_{\mathrm{LY}}$, this correlation function $G(\mathbf{x}; z)$ is expected to have the scaling form

$$G(\mathbf{x}; z = z_{\mathrm{LY}}\,\mathrm{e}^{-\epsilon}) = c\epsilon^{-a} g(b|\mathbf{x}|\epsilon^{\nu}) + \text{higher order terms in } \epsilon \tag{18}$$

where b and c are non-universal, lattice-dependent constants. We choose b such that $g(\xi) = \exp(-\xi)$ for large ξ, and c is fixed by requiring $g(0) = 1$.

The scaling function $g(\xi)$ tends to a constant limiting value as ξ tends to zero, and decreases to zero exponentially fast as ξ tends to infinity. Let the power-series expansion of $g(\xi)$ about $\xi = 0$ be given by

$$g(\xi) = \sum_{k=0}^{\infty} g_k \xi^k. \tag{19}$$

Substituting this in equation (18), we get

$$G(\mathbf{x}; z) \sim \sum_{k=0}^{\infty} g_k b^k |\mathbf{x}|^k \left(1 - \frac{z}{z_{\mathrm{LY}}}\right)^{\nu k - a}. \tag{20}$$

To get $\phi(\mathbf{x}, s)$, we need to determine the coefficient of z^s in the above expansion. From the binomial expansion of $\left(1 - \frac{z}{z_{\mathrm{LY}}}\right)^{\nu k - a}$ we immediately get

$$G(\mathbf{x}; z) \sim \sum_{s=0}^{\infty} \left(\frac{z}{z_{\mathrm{LY}}}\right)^s \sum_{k=0}^{\infty} g_k b^k |\mathbf{x}|^k \frac{\Gamma(a + s - \nu k)}{\Gamma(s+1)\Gamma(a - \nu k)}. \tag{21}$$

For fixed k, for large s, we have

$$\frac{\Gamma(s + a - \nu k)}{\Gamma(s+1)} \to s^{a-1-\nu k}. \tag{22}$$

Using this in equation (21), we get

$$f(|\mathbf{x}|s^{-\nu_\perp}) = \sum_{k=0}^{\infty} g_k \left(\frac{|\mathbf{x}|}{s^{\nu}}\right)^k \frac{1}{\Gamma(a - \nu k)} \tag{23}$$

where the correlation length exponent ν for the HCLG problem is the same as the transverse size exponent $\nu_\perp$ for directed animals.

For $d = 1$, it is easy to determine $G(x; z)$ explicitly. In this case, using equation (13), we get the scaling form for $G(x; z)$ as

$$G(x; \epsilon) = \frac{1}{4\epsilon} \exp(-2x\sqrt{\epsilon}) \tag{24}$$

so that $a = 1$, $\nu = 1/2$, $b = 2$ and $c = 1/4$. The scaling function $g(\xi)$ is simply given by

$$g(\xi) = \exp[-\xi] \tag{25}$$

hence, $g_k = 1/\Gamma[k+1]$. Using this, and the values of a and ν in equation (23), we get the leading singular behaviour of $\phi(x, s)A_s$ on the square lattice to be

$$\phi(x, s)A_s = \frac{1}{4} y_c^{-s} \sum_{k=0}^{\infty} \frac{\left(-\frac{\sqrt{3}x}{\sqrt{s}}\right)^k}{\Gamma(k+1)} \frac{1}{\Gamma\left(1 - \frac{k}{2}\right)}. \tag{26}$$

Since $\Gamma\left(1-\frac{k}{2}\right)$ has poles when k is an even integer, only the odd terms contribute to the sum. The resulting series is easy to sum explicitly, giving

$$\phi(x,s)A_s = \frac{1}{4}3^s \operatorname{erfc}\left(\frac{\sqrt{3}x}{2\sqrt{s}}\right) + \text{correction to scaling terms} \tag{27}$$

where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x$.

This gives $\phi(x,s)$ for large s to be

$$\phi(x,s) = \frac{\sqrt{3\pi s}}{4}\operatorname{erfc}\left(\frac{\sqrt{3}x}{2\sqrt{s}}\right) + \text{correction to scaling terms} \tag{28}$$

From $\phi(x,s)$ we can also derive the expression for the qth transverse moment of the directed animals. The qth transverse moment of a cluster of size s denoted by μ_{qs} is defined as

$$\mu_{qs} = \sum_{i=-s}^{s}\phi(i,s)|i|^q. \tag{29}$$

Using the scaling form of $\phi(i,s)$, for large s, we see that $\left(\mu_{qs}\big/sR_s^q\right)$ is a universal constant. Denoting it by C_q, we get

$$\frac{\mu_{qs}}{sR_s^q} = C_q = \int_{-\infty}^{\infty}\mathrm{d}^d\mathbf{x}\,|\mathbf{x}|^q f(|\mathbf{x}|). \tag{30}$$

For $d = 1$, we get

$$C_q = \frac{1}{q+1}(3)^{\frac{q}{2}}\Gamma\left[1+\frac{q}{2}\right] \tag{31}$$

and $R_s = \frac{2}{3}\sqrt{s}$. Hence, we get the scaling function (16) in $d = 1$ to be

$$f(r) = \frac{\sqrt{\pi}}{2\sqrt{3}}\operatorname{erfc}\left(\frac{r}{\sqrt{3}}\right). \tag{32}$$

In the entire low-density phase of the HCLG, in any dimension d, we expect the correlation function $G(\mathbf{x};z)$ to have an exponential decay at large $|\mathbf{x}|$. But the behaviour of the scaling function $f(r)$ for large r is in general different. Suppose $\ln f(r)$ varies as $-r^\alpha$ for large r. Putting this behaviour in equation (3), and using $A_s \sim \lambda^s$ and equation (12) , we get

$$G(\mathbf{x};z_{\mathrm{LY}}\,\mathrm{e}^{-\epsilon}) \sim \sum_s \exp(-|\mathbf{x}|^\alpha s^{-\nu\alpha} - \epsilon s). \tag{33}$$

For large $|\mathbf{x}|$, the integral can be estimated by the steepest descent, and gives $\log g(\xi)$ varying as $\xi^{\frac{\alpha}{1+\nu\alpha}}$. Since this should be linear in ξ, we see that $\alpha = \frac{1}{1-\nu}$. As a check, we see that in $d = 1$, $\nu = 1/2$, and $f(r)$ varies as $\exp(-r^2)$ for large r.

The case $x = 0$ is special, in that the density–density correlation function $G(0,z)$ is always equal to $\rho(1-\rho)$ for the hard-core lattice gas for any d-dimensional lattice. Hence, if one knows ρ as a function of the activity z (equivalently, in the DA problem, one knows the animal number generating function $A(y)$), then one can determine $\Psi(0;y)$ in terms of $A(y)$ alone. For a $(d+1)$-dimensional DA on a body-centred hypercubical lattice it is given by

$$\Psi(0;y) = \frac{1}{1+y}A(y)[1+A(y)]. \tag{34}$$

For $d = 1$, using the $A(y)$ from equation (6), we get for the square lattice DA problem

$$\Psi(0;y) = \frac{y}{1-2y-3y^2}. \tag{35}$$

Expanding in powers of y we get

$$\phi(0, s) = \frac{3^s + (-1)^{s-1}}{4A_s} \tag{36}$$

where A_s is as given in equation (7). For large s, $\phi(0, s)$ varies as $s^{1/2}$ as expected.

A similar analysis can be extended to higher dimensions. In general, the scaling function $f(r)$ tends to a finite value as r tends to 0, and hence $\phi(0, s)$ varies as $s^{1-\theta}$.

For $d > 7$, mean-field theory becomes asymptotically exact [19] with $R_s \sim s^{1/4}$ and the scaling form of $f(r)$ is

$$f(r) = \frac{2}{\Omega_d} \frac{1}{r^{d-2}} \exp(-r^2) \tag{37}$$

where Ω_d is the surface area of a d-dimensional unit sphere and is equal to $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

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